A CYLINDRICALLY SYMMETRIC SHOCK WAVE IN THE PRESENCE OF DISSIPATIVE PHENOMENA

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There exists a known self-modelling solution (see, e.g. [1 and 2]), of the problem of a cylindrically symmetric shock wave converging towards its axis. Any other solution of this problem in the vicinity of the axis can be regarded as asymptotic to the self-modelling one. The self-modelling solution indicates an unlimited growth of temperature and velocity with decreasing distance from the axis. Therefore, in the space near the axis of an order of magnitude of the mean free path of particles, dissipative phenomena, in the first instance due to viscosity and thermal conductivity, become pronounced.

In the following a solution has been derived for the case of a fully ionized plasma which takes into account these phenomena. It can be treated as a refinement of the self-modelling one.

The proposed solution may be criticized as to its physical meaning, particularly with respect to the structure of the shock wave front, on the grounds that a hydrodynamic approximation is inadequate for a case in which the characteristic scale for the change of magnitudes is of the order of the free path of particles. Nevertheless, one can reasonably expect that the hydrodynamic approach will give a qualitatively correct representation of the phenomena in the vicinity of this axis with viscosity and thermal conductivity effects accounted for, as much as its application was proved under the more stringent conditions in the case of a plane shock wave [3].

1. We shall now consider a system of hydrodynamic equations which take into account fundamental dissipative phenomena, namely: thermal conductivity and viscosity of ions, thermal conductivity of electrons, and the energy exchange between ions and electrons by way of collision. For a cylindrically symmetric wave in a perfect plasma of mass M of ions, ion unit changes, and assuming the adiabatic exponent to be $\gamma = {}^5/_3$, this system of equations is [4]

$$\rho \frac{du}{dt} + \frac{\partial}{\partial r}\left[\frac{k}{M}\rho(T+\Theta)\right] = \frac{\partial}{\partial r}\left(\mu \frac{\partial u}{\partial r}\right) + \mu \frac{\partial}{\partial r}\frac{u}{r} - \frac{1}{2}\frac{u}{r}\frac{\partial \mu}{\partial r} \qquad (1)$$

$$\frac{3}{2}\rho \frac{k}{M}\frac{dT}{dt} - \frac{k}{M}T\frac{d\rho}{dt} = \mu\left(\left(\frac{\partial u}{\partial r}\right)^2 - \frac{u}{r}\left(\frac{\partial u}{\partial r} - \frac{u}{r}\right)\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(r\varkappa_i \frac{\partial T}{\partial r}\right) - Q$$

$$\frac{3}{2}\rho \frac{k}{M}\frac{d\theta}{dt} - \frac{k}{M}\Theta\frac{d\rho}{dt} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\varkappa_e \frac{\partial \Theta}{\partial r}\right) + Q, \qquad \frac{d\rho}{dt} + \rho\frac{1}{r}\frac{\partial}{\partial r}(ru) = 0$$

Here μ is the coefficient of viscosity of ions; $\varkappa_i$ and $\varkappa_e$ are, respectively, the thermal conductivity coefficients of ions and electrons; Q is the rate of energy transfer from ions to electrons, and u and ρ are the velocity and density of plasma. In these equations account has been taken of the different temperatures of the ions and electrons, denoted by T and Θ, respectively. The limits of applicability of these equations to plasma are known [5].

In accordance with the kinetic theory of plasma its dissipation parameters are [6 to 8]

$$\mu = 1.08\frac{M^{1/2}}{e^4L}(kT)^{5/2}, \qquad \varkappa_i = 3.35\frac{k}{m^{1/2}e^4L}\left(\frac{m}{M}\right)^{1/2}(kT)^{5/2}$$
$$\varkappa_e = 1.9\frac{k}{m^{1/2}e^4L}(k\Theta)^{5/2}, \qquad Q = 5.0\frac{m^{1/2}e^4L}{M^3}\rho^2\frac{k(T-\Theta)}{(k\Theta)^{3/2}} \tag{2}$$

The mean free path of ions and electrons is given by

$$l_i = \frac{M(kT)^2}{\rho e^4L}, \qquad l_e = \frac{M(k\Theta)^2}{\rho e^4L} \tag{3}$$

Here L is the Coulomb logarithm, m is the mass of an electron, e is the elementary electric charge, and k the Boltzmann's constant.

$$\mu = \mu_0\left(\frac{kT}{M}\right)^{5/2}, \qquad \varkappa_i = 3.1\mu_0\left(\frac{kT}{M}\right)^{5/2}\frac{k}{M}$$
$$Q = 5.4\left(\frac{m}{M}\right)^{1/2}\mu_0^{-1}\rho^2\frac{k}{M}(T-\Theta)\left(\frac{k\Theta}{M}\right)^{-3/2}$$
$$\varkappa_e = 1.76\left(\frac{M}{m}\right)^{1/2}\mu_0\left(\frac{k\Theta}{M}\right)^{5/2}\frac{k}{M} \tag{4}$$

From (2) we easily find

$$\mu_0 = 1.08\frac{M^3}{e^4L} \tag{5}$$

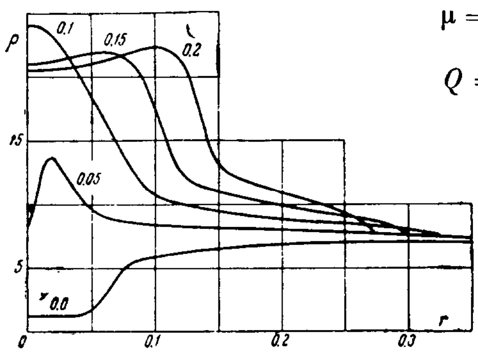

Fig. 1

In order to arrive at quantitative results we shall consider a a deuterium plasma, and assume $(M/m)^{1/2} = 60.5$.

We shall apply system (1) to the solution of a problem, where $\rho = \rho_0$ and $u = T = \Theta = 0$ are given on the segment $0 \leqslant r \leqslant r^\circ$ at the initial instant.

The boundary conditions are

$$u = 0, \qquad \frac{\partial T}{\partial r} = \frac{\partial\Theta}{\partial r} = 0 \quad \text{for } r = 0$$
$$\frac{dr^\circ}{dt} = u, \qquad \frac{k}{M}\rho(T+\Theta) = f(t) \quad \text{for } r = r^\circ \tag{6}$$

where function $f(t)$ is the same as in the self-modelling solution (*)

*) Strictly speaking, the left-hand side of the second boundary condition which represents the radial component of the impulse stream should read $k/M\rho(T+\Theta) - \sigma'_{rr}$, where σ'_{rr} is the component of the viscosity tensor. However, the value of the additional term σ'_{rr} is very small.

If the moment of focusing of the self-modelling wave is taken as the zero time reference, then the self-modelling solution depends only on parameters ρ_0 and ξ_0, where $tr^{-\nu} = \xi_0$ is the equation of the shock wave front, $\nu = \nu(\gamma)$ is the self-modelling exponent which in the case of $\gamma = 5/3$, is $\nu = 1.226$.

Stated in this way, our problem has four independent determining parameters, viz. ρ_0, ξ_0, μ_0 and r°. We shall substitute dimensionless variables, selecting as units the following

$$\rho_0, \quad r^\circ = \left(\frac{\mu_0}{\rho_0 \xi_0^4}\right)^{1/(4\nu-3)}, \quad t_0 = \xi_0 r_0^\nu, \quad u_0 = \frac{r_0}{t_0}, \quad T_0 = \frac{M}{k}\frac{r_0^2}{t_0^2} \qquad (7)$$

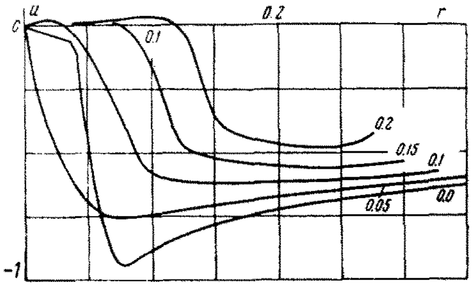

Fig. 2

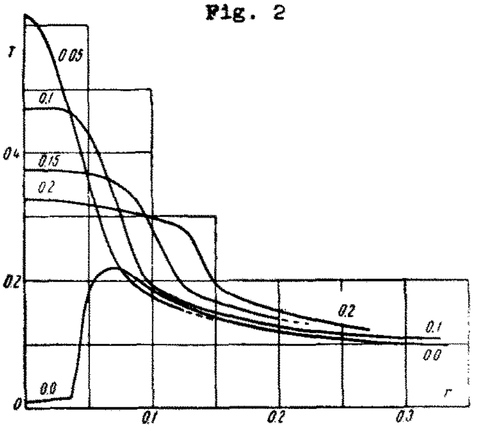

Fig. 3

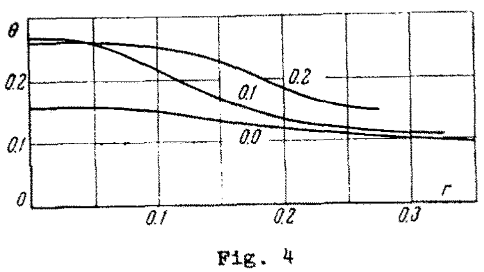

Fig. 4

It will be seen that r° is characteristic of the mean free path, as from (3) and (7) we find that

$$l_i \approx r^\circ T^2 / \rho \qquad (8)$$

Here T and ρ are dimensionless and the factor of the order of 1 has been omitted.

It can be easily proved that the dimensionless functions which we shall again denote by ρ, u, T and θ, must satisfy the system of equations (1) and (4) (with $\mu_0 = k/M = 1$), the initial conditions $u = T = \Theta = 0$ and $\rho = 1$ for $0 \leqslant r \leqslant R = r^\circ / r_0$ and the initial conditions (6) in their dimensionless form. Consequently, there remains in our problem one determining parameter R which may be called the Reynolds Number. Let us consider the influence of this parameter by assigning to it a certain value $R = R_*$. The problem as stated and with boundary conditions as in the self-modelling solution has a meaning, only if at radii $r \sim R_*$ the effects of viscosity and thermal conductivity are insignificant. In other words, it is essential that the shock wave reaches the self-modelling state before any of the dissipative effects begin to take place. But then for $r > R_*$ the solution can also be assumed to be a self-modelling one, as this will lead to the same conditions for $r = R_*$, i.e. any arbitrary selection of $R \geqslant R_*$, will have no effect on the solution for $r < R_*$. Therefore, if the stated problem has any meaning, its solution is independent of parameter R, and it will be sufficient to solve it for $R \geqslant R_*$ once only. The numerical value of R_* can be determined in the following manner. The effects due to viscosity and thermal conductivity become appreciable at distances of the order of l. From Formula (8) we have

$$\frac{l_i}{r^\circ} \approx \frac{1}{R} \frac{T^2}{\rho} \tag{9}$$

These effects will be negligible at radii $\sim r^\circ$, whenever l_i/r° is sufficiently small. With T and ρ taken from the self-modelling solution, we obtain from (9) $R_* \sim 1$.

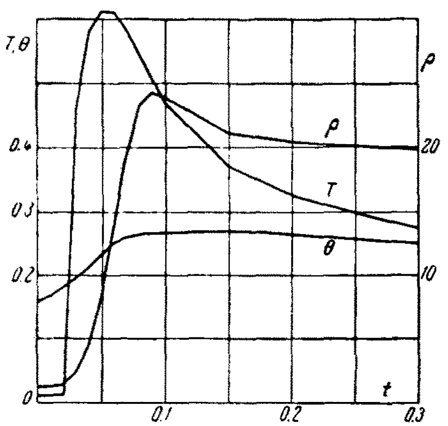

Fig. 5

There exists, therefore, a category of problems concerning converging shock waves in which dissipative effects may alter their solution qualitatively, as compared to the results obtained in the self-modelling solution. A similar situation had been noted in [9] which dealt with the problem of flooding of bubbles in a viscous fluid for which two qualitatively different types of solution were obtained.

Our problem was solved by a numerical method similar to the described in [4]. Some of the results of these computations are given below. There is a great similarity between the self-modelling solution and the solution for a converging shock wave up to the moment at which the distance of the wave front from the axis becomes comparable to the width of the blurred zone of the front, i.e. to the free path length $l_i \approx T^2/\rho$. The most significant differences between the two solutions become apparent after the reflection of the wave from the axis. The distribution and numerical values of ρ, u, T and Θ are shown for this stage as functions of r in Figs. 1 to 4 for moments of time indicated thereon. Fig.5 gives ρ, T and Θ as functions of time.

In the self-modelling solution the density at the moment of focusing ($t = 0$) is throughout equal 7, then it increases up to the moment of arrival of the reflected wave to 11.7, when it jumps to 22.9. At the axis $\rho = 0$, while the mean density in the space between the axis and the reflected wave it is equal to 19.2 at all times.

It will be seen from a comparison of our solution (Fig.1) with the self-modelling one that the greatest difference in densities appears in the vicinity of the axis where $\rho \approx 20$ to 24. At the reflected wave front it is close enough to $\rho \approx 24$ to 22, as given by the self-modelling solution, while the mean density is somewhat greater. The formula for the maximum temperature T_{max} arrived at in the self-modelling solution for a given radius r is $T_{max} = 0.224 r^{-0.452}$, and is reached at the reflected wave front. The corresponding value of $T + \Theta$ calculated for distances up to $r \approx 0.1$ differs significantly from T_{max}. The zone in which $0 < r < 0.1$ may therefore be considered as that for which the proposed solution differs from the self-modelling one. It is of the order of several tens of the free path. Inside it $T + \Theta < T_{max}$, and T and Θ reach their maxima of 0.614 and 0.268, respectively, at the axis.

The proposed method can be applied for the theoretical assessment of temperatures and densities at the focusing point of shock waves not only of an ionized plasma but for other problems involving dissipative phenomena of a different nature.

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